Indian Statistical Institute
B. Math. Hons. I Year
Semestral Examination 2002-2003 (Backpaper)
Algebra II

You may **use** and **quote** any result proved in the **classroom** or in **assignments with out proof**.
Total marks: 55
Section I: Answer any four questions and each questions carries 5 marks.

(1) Let G be a finite group. Then (a) H is a proper subgroup of G implies $G \neq \cup_{g \in G} gHg^{-1}$ and (b) if p is the smallest prime integer dividing the order of G, then any subgroup H of G of index p is normal.

(2) Let T be a linear operator on a hermitian space V. Then T is normal if and only if there exists a polynomial $f \in \mathbb{C}[t]$ such that $f(T) = T^*$.

(3) Let R be a pid and $f \in R[x]$. Suppose $a \in R$ is nilpotent. Then $f(a)$ is a unit if and only if $f(0)$ is a unit.

(4) Show that the ideal $I = (2, x)$ can not be written as a sum of cyclic $\mathbb{Z}[x]$-module.

(5) Let A be an $n \times n$-matrix. Then $A^2 = A$ implies A is similar to a matrix diag $(1, 1, \cdots, 1, 0, \cdots, 0)$.

Section II: Answer all questions

(1) (a) Let F be a field. Then for $a \in F$, show that $(x - a)$ divides $f(x) - f(a)$ for all $f \in F[x]$ and show that $I_a = \{g \in F[x] \mid g(a) = 0\}$ is an ideal and write I_a as principal ideal. Determine all $a \in F$ for which I_a is a maximal ideal.

(b) Let $R = \mathbb{Q}[x]$ and $I = (x^2 + 2)$. Is R/I an integral domain?

Marks: 7+4 =11

(2) (a) Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be a linear operator. Find the possible Jordan forms of T.

(b) Classify upto similarity all 3×3 complex matrices A such that $A^3 = I$.

(c) Let A be a complex 3×3 matrix given by

$$A = \begin{pmatrix} 2 & 0 & 0 \\ a & 2 & 0 \\ b & c & -1 \end{pmatrix}.$$

Find its Jordan form and show that A is diagonalizable if and only if $a = 0$.

Marks: 6+10+8 = 24